Adiabatic Limits, Vanishing Theorems and the Noncommutative Residue

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Abstract

In this paper, we compute the adiabatic limit of the scalar curvature and prove several vanishing theorems, we also derive a Kastler-Kalau-Walze type theorem for the noncommutative residue in the case of foliations.

Keywords: Foliations; adiabatic limits; vanishing theorems; noncommutative residue

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1 Introduction

Let (M, F) be a compact foliated manifold. If F is spin and there is a metric g^F on F such that the scalar curvature of g^F is positive, then $\widehat{A}$-genus of M vanishes as proved in [Co]. A. Connes proved this theorem by a highly noncommutative method. In [LZ], Liu and Zhang used adiabatic limits to study foliated manifolds. They constructed the sub-Dirac operator associated to foliations with spin leave and presented a direct geometric proof of the vanishing theorem of Connes for almost Riemannian foliations by taking the adiabatic limit of the square of the sub-Dirac operator. In [LMZ], the author used the sub-Dirac operator to prove rigidity theorems for foliations. In [Ko2], [LK], adiabatic limits were used to study the spectral geometry for Riemannian foliations. In [Ru], Rumin studied the adiabatic limits of some geometric objects for contact manifolds.

Here are some motivations of this paper.

1) For contact manifolds and in general for a manifold M with splitting tangent bundle $TM = F \oplus F^{\perp}$ and F is not integrable, we hope to get a Connes type vanishing theorem by taking adiabatic limits.

2) For complex foliations, by using the Bismut-Kodiara-Nakano formula in [Bi] and taking adiabatic limits, we hope to get a vanishing theorem for foliation.

3) The Kastler-Kalau-Walze theorem says that the noncommutative residue (see [Wo], [FGV]) of the $-\mathrm{dim}M+2$ power of the Dirac operator for even-dimensional spin manifolds M is proportional to the Einstein-Hilbert action. For a foliation (M, F) with spin leave, by using the sub-Dirac operator in [LZ] and considering the adiabatic limit of the noncommutative residue of the $-\mathrm{dim}M+2$ power of the sub-Dirac operator, we hope to derive a Kastler-Kalau-Walze type theorem for foliations.

In this paper, We define $\Phi(\omega)$, A, B, Ψ by (2.12), (2.20), (3.24), (4.33) in the following respectively. We will prove the following theorems.

Theorem I *Let (M, F, g^F) be a compact and transversally oriented foliation with spin leave, if $A(\omega, \phi_i(F^\perp)) > 0$ for any $\phi_i(F^\perp)$ appeared in (2.22), then $< \widehat{A}(TM), [M] >= 0$.*

Theorem II *Let M be a compact oriented manifold, if 1) $TM = F \oplus F^\perp$ and F may not be integrable and oriented spin, 2) $B > 0$, then $< \widehat{A}(TM), [M] >= 0$.*

Theorem III *Let (M, F) be a compact complex foliation. If $\Psi > 0$, then the Euler number $\mathrm{Eul}(\xi)$ of the holomorphic bundle ξ vanishes.*

Theorem IV *Let (M^n, F) be a compact even-dimensional oriented foliation with spin leave and codimension q, and D_F be the sub-Dirac operator, then $\lim_{\varepsilon\to 0}\varepsilon^{\frac{q}{2}}\mathrm{Res}(D_{F,\varepsilon}^{-n+2})$ is proportional to $\int_M [k^F + \Phi(\omega)]\mathrm{dvol}_g$.*

This paper is organized as follows: In Section 2, we compute the adiabatic limit of the scalar curvature explicitly by following the method in [LZ], then by using this result we give a new vanishing theorem for general foliations with spin leave. In Section 3, for a manifold M with splitting tangent bundle $TM = F \oplus F^\perp$ and F may not be integrable, we also compute the adiabatic limit of the scalar curvature similarly, and we note that some extra singular terms $O(\frac{1}{\varepsilon})$ will appear. We consider the adiabatic limit of εD_F^2 and derive a vanishing theorem. In Section 4, by the Bismut-Kodiara-Nakano formula and taking adiabatic limits, a vanishing theorem can be obtained for complex foliations. In Section 5, a formula similar to the Kastler-Kalau-Walze theorem for foliations with spin leave is given.

2 Vanishing theorem for foliations

First we recall the basic setup and some facts in [LZ] (for details, see [LZ]).

Let (M, F) be a foliation, that is, F is an integrable sub-bundle of the tangent bundle TM. Take a metric g^{TM} on TM as in [LZ], then

$$TM = F \oplus F^\perp;\ g^{TM} = g^F \oplus g^{F^\perp}, \tag{2.1}$$

where $F^\perp$ is the orthogonal complement of F in TM with respect to g^{TM} and g^F (resp. $g^{F^\perp}$) the metric on F (resp. $F^\perp$). Let p, $p^\perp$ be the orthogonal projection from TM to F, $F^\perp$. Let ∇^{TM} be the Levi-Civita connection of g^{TM} and ∇^F (resp.$\nabla^{F^\perp}$) be the restriction of ∇^{TM} to F (resp. $F^\perp$). For any $\varepsilon > 0$. let $g^{TM,\varepsilon}$ be the metric

$$g^{TM,\varepsilon} = g^F \oplus \frac{1}{\varepsilon} g^{F^\perp}. \tag{2.2}$$

Let $\nabla^{TM,\varepsilon}$ be the Levi-Civita connection of $g^{TM,\varepsilon}$ and $\nabla^{F,\varepsilon}$ (resp.$\nabla^{F^\perp,\varepsilon}$) be the restriction of $\nabla^{TM,\varepsilon}$ to F (resp. $F^\perp$). Then we have formulas (1.5)-(1.8) in [LZ]. Let $\dot{\nabla}$ be the Bott connection on $F^\perp$ and $\dot{\nabla}^\star$ be the dual connection of $\dot{\nabla}$ and

$$\omega := \dot{\nabla}^\star - \dot{\nabla};\ \widehat{\nabla} = \frac{\dot{\nabla} + \dot{\nabla}^\star}{2}. \tag{2.3}$$

Let $k^{TM,\varepsilon}$ be the scalar curvature associated to $\nabla^{TM,\varepsilon}$. In the following, we will compute the adiabatic limit $\lim_{\varepsilon\to 0}(k^{TM,\varepsilon})$.

Let $\{f_i\}_{i=1}^p, \{h_s\}_{s=1}^q$ be an orthonormal basis of $g^{TM} = g^F \oplus g^{F^\perp}$. By (2.32) in [LZ], we get

$$\lim_{\varepsilon\to 0}\langle R^{TM,\varepsilon}(f_i,f_j)f_i,f_j\rangle = \langle R^F(f_i,f_j)f_i,f_j\rangle. \tag{2.4}$$

By (2.35) in [LZ], for $X \in \Gamma(F),\ U,V \in \Gamma(F^\perp)$,

$$A := \lim_{\varepsilon\to 0}p^\perp\nabla_U^{TM,\varepsilon}X = \frac{1}{2}\sum_{s=1}^q \omega(X)(U,h_s)h_s. \tag{2.5}$$

By (2.5) and (2.33), (1.13) in [LZ], we get

$$\begin{aligned}\lim_{\varepsilon\to 0}\langle R^{TM,\varepsilon}(X,U)X,U\rangle &= -\frac{1}{2}\omega(p\nabla_X^{TM}X)(U,U) - \frac{1}{2}\omega(X)(U,p^\perp[X,U])\\ &\quad+\langle[X,A],U\rangle + \frac{1}{2}\omega(X)(U,A).\end{aligned} \tag{2.6}$$

By (1.7), (1.8), (1.6) and (1.13) in [LZ], we have

$$\begin{aligned}\langle\nabla_V^{TM,\varepsilon}\nabla_U^{TM,\varepsilon}U,V\rangle &= \langle\nabla_V^{TM,\varepsilon}(p\nabla_U^{TM,\varepsilon}U),V\rangle + \langle p^\perp\nabla_V^{TM,\varepsilon}p^\perp\nabla_U^{TM,\varepsilon}U,V\rangle\\ &= -\frac{1}{2}\langle p\nabla_U^{TM,\varepsilon}U, 2\nabla_V^{TM}V\rangle + \frac{\varepsilon}{2}\langle p\nabla_U^{TM,\varepsilon}U,[V,V]\rangle + \langle\nabla_V^{F^\perp}\nabla_U^{F^\perp},V\rangle\\ &= -\langle p\nabla_U^{TM,\varepsilon}U,\nabla_V^{TM}V\rangle + \langle\nabla_V^{F^\perp}\nabla_U^{F^\perp},V\rangle\\ &= -\langle\nabla_U^{TM}U,p\nabla_V^{TM}V\rangle + \frac{1}{2}\langle p\nabla_V^{TM}V,2\nabla_U^{TM}U\rangle - \frac{1}{2\varepsilon}\langle p\nabla_V^{TM}V,2\nabla_U^{TM}U\rangle + \langle\nabla_V^{F^\perp}\nabla_U^{F^\perp},V\rangle\\ &= \frac{1}{2\varepsilon}\omega(p\nabla_V^{TM}V)(U,U) + O(1).\end{aligned} \tag{2.7}$$

Similarly, we have,

$$\begin{aligned}\langle p\nabla_V^{TM,\varepsilon}U,\nabla_U^{TM}V+\nabla_V^{TM}U\rangle &= \langle\nabla_V^{TM,\varepsilon}U,p(\nabla_U^{TM}V+\nabla_V^{TM}U)\rangle\\ &= \frac{1}{2\varepsilon}\langle p(\nabla_U^{TM}V+\nabla_V^{TM}U),\nabla_U^{TM}V+\nabla_V^{TM}U\rangle + O(1)\\ &= -\frac{1}{2\varepsilon}\omega(p(\nabla_U^{TM}V+\nabla_V^{TM}U))(U,V) + O(1).\end{aligned} \tag{2.8}$$

By (2.7), (2.8) and (2.34) in [LZ], we get,

$$\lim_{\varepsilon\to 0}\varepsilon\langle R^{TM,\varepsilon}(U,V)U,V\rangle = \frac{1}{4}\omega(p(\nabla_U^{TM}V+\nabla_V^{TM}U))(U,V) - \frac{1}{2}\omega(p\nabla_V^{TM}V)(U,U). \tag{2.9}$$

By definition,

$$-k^{TM,\varepsilon} = \sum_{i,j=1}^{p} \langle R^{TM,\varepsilon}(f_i, f_j)f_i, f_j\rangle + \sum_{s,t=1}^{q} \varepsilon\langle R^{TM,\varepsilon}(h_s, h_t)h_s, h_t\rangle$$

$$+2\sum_{i=1}^{p}\sum_{s=1}^{q}\langle R^{TM,\varepsilon}(f_i, h_s)f_i, h_s\rangle. \tag{2.10}$$

By (2.4), (2.6), (2.9), (2.10), then we get

$$\lim_{\varepsilon\to 0}(k^{TM,\varepsilon}) = k^F + \Phi(\omega), \tag{2.11}$$

where

$$\Phi(\omega) = \sum_{s,t=1}^{q}[-\frac{1}{4}\omega(p(\nabla^{TM}_{h_s}h_t + \nabla^{TM}_{h_t}h_s))(h_s, h_t) + \frac{1}{2}\omega(p\nabla^{TM}_{h_t}h_t)(h_s, h_s)]$$

$$+\sum_{i=1}^{q}\sum_{s=1}^{q}\{\frac{1}{2}\omega(p\nabla^{TM}_{f_i}f_i)(h_s, h_s) + \frac{1}{2}\omega(f_i)(f_i, p^{\perp}[f_i, h_s])$$

$$-\langle[f_i, A], h_s\rangle - \frac{1}{2}\omega(f_i)(h_s, A)\}. \tag{2.12}$$

Borrowing the idea in [LZ], we will prove a vanishing theorem. By Theorem 1.1 in [LZ], we can write for $Y \in \Gamma(F), U \in \Gamma(F^{\perp})$,

$$\nabla^{F^{\perp},\varepsilon}_Y U = \widehat{\nabla_Y}U + O(\varepsilon) = \dot{\nabla}_Y U + \frac{\omega(Y)U}{2} + O(\varepsilon). \tag{2.13}$$

then because the curvature of the Bott connection vanishes along leaves, we have for $X, Y \in \Gamma(F), U, V \in \Gamma(F^{\perp})$,

$$\langle R^{F^{\perp},\varepsilon}(X, Y)U, V\rangle = \langle \widehat{R}^{F^{\perp}}(X, Y)U, V\rangle + O(\varepsilon)$$

$$= \langle \dot{R}(X, Y)U, V\rangle + \frac{1}{2}\langle\{\dot{\nabla}_X\omega(Y) - \dot{\nabla}_Y\omega(X)\}U, V\rangle$$

$$+\frac{1}{4}\langle\{[\omega(X), \omega(Y)] - 2\omega([X, Y])\}U, V\rangle + O(\varepsilon)$$

$$= \frac{1}{2}\langle\{\dot{\nabla}_X\omega(Y) - \dot{\nabla}_Y\omega(X)\}U, V\rangle + \frac{1}{4}\langle\{[\omega(X), \omega(Y)] - 2\omega([X, Y])\}U, V\rangle + O(\varepsilon). \tag{2.14}$$

By $\nabla^{F^{\perp},\varepsilon}_V U = \nabla^{F^{\perp}}_V U$ and (2.13), similar to (2.14), for $X \in \Gamma(F), U, V, Z, Z_1, Z_2 \in \Gamma(F^{\perp})$ we have

$$\langle R^{F^{\perp},\varepsilon}(X, U)V, Z\rangle = O(1); \langle R^{F^{\perp},\varepsilon}(U, V)Z_1, Z_2\rangle = O(1). \tag{2.15}$$

So by (2.14), (2.15), the sum of the last three terms in (2.30) in [LZ] is

$$= \frac{1}{8}\sum_{i,j=1}^{p}\sum_{s,t=1}^{q}\langle \widehat{R}^{F^{\perp}}(f_i, f_j)h_t, h_s\rangle c(f_i)c(f_j)\widehat{c}(\sqrt{\varepsilon}h_s)\widehat{c}(\sqrt{\varepsilon}h_t) + O(\sqrt{\varepsilon}). \tag{2.16}$$

In order to prove a vanishing theorem, we will estimate the norm of $\widehat{c}(\sqrt{\varepsilon}h_t)$ under the metric g^ε. By

$$||\widehat{c}(\sqrt{\varepsilon}h_t)||_{g^\varepsilon} = ||\widehat{c}(h_t)||_g, \tag{2.17}$$

so we do not consider the order of ε from $\widehat{c}(\sqrt{\varepsilon}h_t)$. For other three terms including $R^{\phi(F^\perp),\varepsilon}$ in (2.30) in [LZ], similar to (2.16), their sum is

$$\frac{1}{2}\sum_{i,j=1}^{p} c(f_i)c(f_j)\widehat{R}^{\phi(F^\perp)}(f_i,f_j) + O(\sqrt{\varepsilon}). \tag{2.18}$$

Then by (2.30) in [LZ], (2.11),(2.16) and (2.18), we get

$$D^2_{F,\phi(F^\perp),\varepsilon} = -\triangle^{F,\phi(F^\perp),\varepsilon} + k^F + \Phi(\omega)$$

$$+\frac{1}{8}\sum_{i,j=1}^{p}\sum_{s,t=1}^{q}\langle \widehat{R}^{F^\perp}(f_i,f_j)h_t,h_s\rangle c(f_i)c(f_j)\widehat{c}(\sqrt{\varepsilon}h_s)\widehat{c}(\sqrt{\varepsilon}h_t)$$

$$+\frac{1}{2}\sum_{i,j=1}^{p} c(f_i)c(f_j)\widehat{R}^{\phi(F^\perp)}(f_i,f_j) + O(\sqrt{\varepsilon}). \tag{2.19}$$

Let x be a point in M and $||L||_x$ be the norm of the operator L on $(S(F)\widehat{\otimes}\wedge(F^{\perp,\star})\otimes \phi(F^\perp), g_x)$. Let

$$A(\omega,\phi(F^\perp))(x) = \frac{(k^F+\Phi(\omega))(x)}{4} - ||\frac{1}{2}\sum_{i,j=1}^{p} c(f_i)c(f_j)\widehat{R}^{\phi(F^\perp)}(f_i,f_j)||_x$$

$$-||\frac{1}{8}\sum_{i,j=1}^{p}\sum_{s,t=1}^{q}\widehat{R}^{F^\perp}\langle(f_i,f_j)h_t,h_s\rangle c(f_i)c(f_j)\widehat{c}(h_s)\widehat{c}(h_t)||_x. \tag{2.20}$$

By (2.17), (2.19), (2.20), if $A(\omega,\phi(F^\perp))>0$, for sufficiently small $\varepsilon>0$, we get

$$D^2_{F,\phi(F^\perp),\varepsilon} > 0 \text{ and } \widehat{A}(F)L(F^\perp)\mathrm{ch}(\phi(F^\perp)) = 0. \tag{2.21}$$

Let

$$\langle \widehat{A}(TM),[M]\rangle = \sum c_i\langle \widehat{A}(F)L(F^\perp)\mathrm{ch}(\phi_i(F^\perp)),[M]\rangle, \tag{2.22}$$

where c_i is a constant. By (2.21) and (2.22), we have

Theorem 2.1 *Let (M,F,g^F) be a compact and transversally oriented foliation with spin leave, if $A(\omega,\phi_i(F^\perp))>0$ for any $\phi_i(F^\perp)$ appeared in (2.22), then $<\widehat{A}(TM),[M]>=0$.*

Remark. When (M,F) is a Riemannian foliation, then $A=\frac{k^F}{4}$, so we get the vanishing theorem of Connes. As in [LZ], we can get the vanishing theorem of Connes for almost Riemannian foliations from Theorem 2.1.

3 A vanishing theorem when F is not integrable

Recall that for a contact manifold M (for definition, see [Bl]), we have $TM = F \oplus F^{\perp}$ where $F^{\perp}$ is a line bundle and F is not integrable. In order to get a vanishing theorem as in Section 2 for contact manifolds, we need to compute the adiabatic limit of the scalar curvature in this case.

Let (M, g^{TM}) be an oriented Riemannian manifold. Assume that

$$TM = F \oplus F^{\perp},\ g^{TM} = g^F \oplus g^{F^{\perp}}, \tag{3.1}$$

where F may not be integrable. Now we compute the adiabatic limit of the scalar curvature in this case. We use the same notations as in Section 2. Then we have similar formulas to (1.5)-(1.8) in [LZ]. For $\widehat{X} \in TM, X, Y, Z \in \Gamma(F), U, V \in \Gamma(F^{\perp})$,

$$\langle \nabla^{F,\varepsilon}_{\widehat{X}} Y, Z\rangle_{g^{TM}} = \langle \nabla^{F}_{\widehat{X}} Y, Z\rangle_{g^{TM}} + \frac{1}{2}(1-\frac{1}{\varepsilon})\langle p^{\perp}[Y,Z], \widehat{X}\rangle_{g^{TM}}. \tag{3.2}$$

Especially, when $\widehat{X} = X \in \Gamma(F)$,we have

$$\nabla^{F,\varepsilon}_X = \nabla^F_X. \tag{3.3}$$

Furthermore,

$$\langle \nabla^{TM,\varepsilon}_X U, Y\rangle_{g^{TM}} = \langle \nabla^{TM}_X U, Y\rangle_{g^{TM}} + \frac{1}{2}(1-\frac{1}{\varepsilon})\langle [X,Y], U\rangle_{g^{TM}}, \tag{3.4}$$

$$\langle \nabla^{TM,\varepsilon}_V U, X\rangle_{g^{TM}} = \langle \nabla^{TM}_V U, X\rangle_{g^{TM}} - \frac{1}{2}(1-\frac{1}{\varepsilon})\langle X, \nabla^{TM}_V U + \nabla^{TM}_U V\rangle, \tag{3.5}$$

and

$$\langle \nabla^{TM,\varepsilon}_X Y, U\rangle_{g^{TM}} = \varepsilon\langle \nabla^{TM}_X Y, U\rangle_{g^{TM}} + \frac{1}{2}(1-\varepsilon)\langle [X,Y], U\rangle_{g^{TM}}. \tag{3.6}$$

Especially when $X = Y$, we have

$$p^{\perp}\nabla^{TM,\varepsilon}_X X = \varepsilon p^{\perp}\nabla^{TM}_X X. \tag{3.7}$$

Furthermore,

$$\langle \nabla^{TM,\varepsilon}_V Y, U\rangle_{g^{TM}} = -\frac{1}{2}\langle Y, \nabla^{TM}_V U + \nabla^{TM}_U V\rangle + \frac{\varepsilon}{2}\langle Y, [U,V]\rangle, \tag{3.8}$$

$$\nabla^{F^{\perp},\varepsilon}_V = \nabla^{F^{\perp}}_V, \tag{3.9}$$

and

$$\langle \nabla^{F^{\perp},\varepsilon}_X U, V\rangle_{g^{TM}} = \langle [X,U], V\rangle - \frac{1}{2}\langle X, \nabla^{TM}_V U + \nabla^{TM}_U V\rangle - \frac{\varepsilon}{2}\langle X, [U,V]\rangle. \tag{3.10}$$

As in Section 1 in [LZ], we can still define the Bott connection $\dot{\nabla}$ which may not be flat along leave, its dual connection $\dot{\nabla}^{\star}$ and $\omega, \widehat{\nabla}$. Then we still have for $X \in \Gamma(F)$

$$\lim_{\varepsilon \to 0} \nabla^{F^{\perp},\varepsilon}_X = \widehat{\nabla}_X. \tag{3.11}$$

In our case, we denote the scalar curvature (resp. curvature) by $\overline{k}^{TM,\varepsilon}$ (resp. $\overline{R}^{TM,\varepsilon}$) associated to g^ε. We still denote the curvature by $R^{TM,\varepsilon}$ when F is integrable. That is, we just use the expression of $R^{TM,\varepsilon}$ in [LZ]. By (2.32) in [LZ] and (3.2), (3.3), (3.6), we have

$$\langle\overline{R}^{TM,\varepsilon}(f_i,f_j)f_i,f_j\rangle=\langle R^{TM,\varepsilon}(f_i,f_j)f_i,f_j\rangle+(-\frac{1}{2}-\frac{\varepsilon}{4}+\frac{3}{4\varepsilon})\langle p^\perp[f_i,f_j],[f_i,f_j]\rangle_{g^{TM}}. \tag{3.13}$$

By (2.33) in [LZ] and (3.2), (3.7), (3.8), (3.9), (3.10), for $X\in\Gamma(F), U\in\Gamma(F^\perp)$, we have

$$\begin{aligned}\langle\overline{R}^{TM,\varepsilon}(X,U)X,U\rangle&=\langle R^{TM,\varepsilon}(X,U)X,U\rangle+\frac{1}{2}(1-\varepsilon)\langle[X,p\nabla_X^{TM}U],U\rangle\\&+\frac{1}{2}(1-\varepsilon)\langle[X,p\nabla_U^{TM,\varepsilon}X],U\rangle-\frac{1}{2}(1-\varepsilon)\langle[p[X,U],X],U\rangle.\end{aligned}\tag{3.14}$$

By (2.34) in [LZ] and (3.8), (3.9), (3.10), for $U,V\in\Gamma(F^\perp)$, we have

$$\langle\overline{R}^{TM,\varepsilon}(U,V)U,V\rangle=\langle R^{TM,\varepsilon}(U,V)U,V\rangle. \tag{3.15}$$

By (2.32) in [LZ] and (3.13), when $\varepsilon\to 0$, we have

$$\langle\overline{R}^{TM,\varepsilon}(f_i,f_j)f_i,f_j\rangle\sim\langle R^F(f_i,f_j)f_i,f_j\rangle-\frac{1}{2}\langle p^\perp[f_i,f_j],[f_i,f_j]\rangle_{g^{TM}}+O(\frac{1}{\varepsilon}) \tag{3.16}$$

and

$$\lim_{\varepsilon\to0}\varepsilon\langle\overline{R}^{TM,\varepsilon}(f_i,f_j)f_i,f_j\rangle=\frac{3}{4}\langle p^\perp[f_i,f_j],[f_i,f_j]\rangle_{g^{TM}}. \tag{3.17}$$

So

$$\sum_{1\leq i,j\leq p}\lim_{\varepsilon\to0}\varepsilon\langle\overline{R}^{TM,\varepsilon}(f_i,f_j)f_i,f_j\rangle=\frac{3}{4}\sum_{1\leq i,j\leq p}||p^\perp[f_i,f_j]||^2, \tag{3.18}$$

which is globally defined and is positive when F is not integrable. By (2.33), (2.35) in [LZ] ((2.35) is still correct in this case) and (3.14), we have

$$\begin{aligned}&\lim_{\varepsilon\to0}\varepsilon\langle\overline{R}^{TM,\varepsilon}(X,U)X,U\rangle\\&=\lim_{\varepsilon\to0}\left[\varepsilon\langle R^{TM,\varepsilon}(X,U)X,U\rangle+\frac{\varepsilon}{2}(1-\varepsilon)\langle[X,p\nabla_U^{TM,\varepsilon}X],U\rangle\right]\\&=\lim_{\varepsilon\to0}\Bigg[\frac{\varepsilon^2}{2}\langle X,[U,p^\perp\nabla_U^{TM,\varepsilon}X]\rangle+\varepsilon\langle[X,p^\perp\nabla_U^{TM,\varepsilon}X],U\rangle\\&-\frac{\varepsilon}{2}\langle X,\nabla^{TM}_{p^\perp\nabla_U^{TM,\varepsilon}X}U+\nabla_U^{TM}p^\perp\nabla_U^{TM,\varepsilon}X\rangle+\frac{\varepsilon}{2}(1-\varepsilon)\langle[X,p\nabla_U^{TM,\varepsilon}X],U\rangle\Bigg]\\&=\lim_{\varepsilon\to0}\frac{\varepsilon}{2}(1-\varepsilon)\langle[X,p\nabla_U^{TM,\varepsilon}X],U\rangle\end{aligned}\tag{3.19}$$

By (3.2), we get

$$\langle[X,p\nabla_U^{TM,\varepsilon}X],U\rangle=\langle\nabla_U^{TM}X,p\nabla_X^{TM}U\rangle+\frac{1}{2}(1-\frac{1}{\varepsilon})\langle[X,p\nabla_X^{TM}U],U\rangle-\langle\nabla^{TM}_{p\nabla_U^{TM,\varepsilon}X}X,U\rangle \tag{3.20}$$

and

$$p\nabla_U^{TM,\varepsilon}X = \sum_{j=1}^{p}\langle\nabla_U^{TM,\varepsilon}X, f_j\rangle f_j = p\nabla_U^{TM}X + \frac{1}{2}(1-\frac{1}{\varepsilon})\sum_{j=1}^{p}\langle[X,f_j],U\rangle f_j. \tag{3.21}$$

by (3.19-3.21), we have

$$\lim_{\varepsilon\to 0}\varepsilon\langle\overline{R}^{TM,\varepsilon}(X,U)X,U\rangle = -\frac{1}{4}\langle[X,p\nabla_X^{TM}U],U\rangle + \frac{1}{4}\langle\nabla^{TM}_{\sum_{j=1}^{p}\langle[X,f_j],U\rangle f_j}X,U\rangle. \tag{3.22}$$

By (2.34) in [LZ] and (3.15), we get

$$\lim_{\varepsilon\to 0}\varepsilon^2\langle\overline{R}^{TM,\varepsilon}(U,V)U,V\rangle = 0 \tag{3.23}$$

By (3.17), (3.22) and (3.23), we have

$$4B := \lim_{\varepsilon\to 0}\varepsilon\overline{k}^{TM,\varepsilon} = -\frac{3}{4}\sum_{1\leq i,j\leq p}||p^{\perp}[f_i,f_j]||^2$$

$$-\frac{1}{2}\sum_{i=1}^{p}\sum_{s=1}^{q}\{\nabla^{TM}_{\sum_{j=1}^{p}\langle[f_i,f_j],h_s\rangle f_j}f_i,h_s\rangle - \langle[f_i,p\nabla^{TM}_{f_i}h_s],h_s\rangle\}$$

$$= -\frac{3}{4}\sum_{1\leq i,j\leq p}||p^{\perp}[f_i,f_j]||^2 - \frac{1}{2}\sum_{i=1}^{p}\sum_{s=1}^{q}||p\nabla^{TM}_{f_i}h_s||^2 + \frac{1}{2}\sum_{i=1}^{p}\sum_{j=1}^{p}||p^{\perp}\nabla^{TM}_{f_j}f_i||^2, \tag{3.24}$$

which is globally defined and vanishes when F is integrable. By (2.30) in [LZ] and (3.11), similar to Section 2, we have

$$\varepsilon D^2_{F,\phi(F^{\perp}),\varepsilon} = -\varepsilon\Delta^{F,\phi(F^{\perp}),\varepsilon} + \frac{\varepsilon\overline{k}^{TM,\varepsilon}}{4} + O(\varepsilon). \tag{3.25}$$

By (3.24) and (3.25) and that $-\varepsilon\Delta^{F,\phi(F^{\perp}),\varepsilon}$ is nonnegative, we get for sufficient small $\varepsilon>0$, if $B>0$, then $D^2_{F,\phi(F^{\perp}),\varepsilon}>0$. So similarly to Theorem 2.1, we have

Theorem 3.1 *Let M be a compact oriented manifold, if 1) $TM = F\oplus F^{\perp}$ and F is not integrable and oriented spin, 2) $B>0$, then $<\widehat{A}(TM),[M]>=0$.*

For many cases, $TM = F^{\perp}\oplus F$ and $F^{\perp}$ is not integrable. When M be an compact contact metric manifold and $\dim M\geq 3$, then M has a canonical spin^c structure (for details, see [Pe]). In this case, $TM = L\oplus L^{\perp}$ and $L^{\perp}$ is not integrable. Let $g^{TM,\varepsilon} = g^L\oplus\frac{1}{\varepsilon}g^{L^{\perp}}$. Let $D^{\widetilde{L}}$ be the spin^c Dirac operator associated to the Levi-Civita connection and the general complex determine line bundle $\widetilde{L}$. By [LM], we have

$$(D^{\widetilde{L},\varepsilon})^2 = -\Delta^{\widetilde{L},\varepsilon} + \frac{\overline{k}^{TM,\varepsilon}}{4} + \frac{\sqrt{-1}}{2}\Omega^{\widetilde{L}}. \tag{3.26}$$

By (3.26), similar to the proof of Theorem 3.1, we have

Theorem 3.2 *For the above M and $D^{\widetilde{L}}$, if $B>0$, then* $\mathrm{ker}D^{\widetilde{L},\varepsilon}=0$ *for sufficient small $\varepsilon>0$.*

Similarly, let M^{2n} be an compact oriented manifold. We assume $TM=F\oplus F^{\perp}$ and F is not integrable. Let $D_{\varepsilon}=d_{\varepsilon}+\delta_{\varepsilon}$ be the de-Rham Hodge operator (resp. signature operator) associated to (M,g^{ε}). If $B>0$, then the Euler number (resp. signature) of M is zero.

4 A vanishing theorem for complex foliations

Let M be an compact connected complex manifold of complex dimension n with a complex foliated structure of complex dimension p. That is, M is the disjoint union of its complex submanifolds of complex dimension p which locally are defined by $dz_{p+1}=\cdots=dz_{p+q}=0$, where $p+q=n$ and $z_1=x_1+iy_1,\cdots,z_n=x_n+iy_n$ is the complex coordinate of M. We consider M as an almost complex manifold with the canonical almost complex structure J :

$$J(\frac{\partial}{\partial x_k})=\frac{\partial}{\partial y_k};\ J(\frac{\partial}{\partial y_k})=-\frac{\partial}{\partial x_k}. \tag{4.1}$$

Let TM be the holomorphic tangent bundle on M and let $T_{\mathbf{R}}M$ be the real tangent bundle of M as an real manifold. Let F be the real tangent bundle of complex leave and locally

$$F=\mathrm{span}_{\mathbf{R}}\{\frac{\partial}{\partial x_1},\frac{\partial}{\partial y_1},\cdots,\frac{\partial}{\partial x_p},\frac{\partial}{\partial y_p}\}, \tag{4.2}$$

then $J|_F:F\to F$ is a complex structure of the bundle F. As in Section 2, we take $g^{T_{\mathbf{R}}M}=g^F\oplus g^{F^{\perp}}$ and $T_{\mathbf{R}}M/F\cong F^{\perp}$. Since $J:F\oplus F^{\perp}\to F\oplus F^{\perp}$ and $J|_F:F\to F$ are isomorphism, then $J:F^{\perp}\to F^{\perp}$ is an isomorphism, so J is also a complex structure of $F^{\perp}$. We take a positive definite Hermitian structure (F,J,H^F) (resp. $(F^{\perp},J,H^{F^{\perp}})$) of (F,J) (resp. $(F^{\perp},J)$) such that

$$g^F=\mathrm{Re}H^F;\ g^{F^{\perp}}=\mathrm{Re}H^{F^{\perp}}. \tag{4.3}$$

Here $\mathrm{Re}H$ (resp. $\mathrm{Im}H$) denotes the real part (resp. imaginary part) of the Hermitian metric H. Let ω_1 (resp. ω_2) be the Kähler form associated to H^F (resp. $H^{F^{\perp}}$). Then for $X_1,Y_1\in F,\ X_2,Y_2\in F^{\perp}$,

$$\omega_1(X_1,Y_1)=g^F(X_1,JY_1);\ \omega_2(X_2,Y_2)=g^{F^{\perp}}(X_2,JY_2). \tag{4.4}$$

For any $\varepsilon>0$, we define a positive definite Hermitian structure H^{ε} on $T_{\mathbf{R}}M$ by

$$H^{\varepsilon}=H^F\oplus\frac{1}{\varepsilon}H^{F^{\perp}}, \tag{4.5}$$

Then

$$g^{T_{\mathbf{R}}M,\varepsilon}=\mathrm{Re}H^{\varepsilon};\ \widehat{\omega}^{\varepsilon}=\omega_1\oplus\frac{1}{\varepsilon}\omega_2, \tag{4.6}$$

and we write $\widehat{\omega}=\widehat{\omega}^1$.

Now, we recall the Bismut-Kodaira-Nakano formula (for details, see Section 2 in [Bi]). Let ξ be a holomorphic Hermitian vector bundle of complex dimension l. Let ∇^ξ be the holomorphic Hermitian connection on ξ, whose curvature is denoted by $(\nabla^\xi)^2$. $\wedge T^{\star(0,1)}M$ denotes the algebra of forms of type $(0,p)$ $(0\leq p\leq n)$. Let $\overline{\partial}$ be the Dolbeault operator acting on the set Γ of smooth sections of $\wedge T^{\star(0,1)}M\otimes\xi$ equipped with the natural L^2-Hermitian product. Let $\overline{\partial}^\star$ be the formal adjoint of $\overline{\partial}$ with respect to this Hermitian metric. Let ∇^{TM} be the holomorphic Hermitian connection on TM associated to the hermitian metric g^{TM} induced by $g^{T_{\mathbf{R}}M}=g^F\oplus g^{F^\perp}$. Let ω be the Kähler form associated to $H=H^F\oplus H^{F^\perp}$. Locally,

$$\wedge T^{\star(0,1)}M\otimes\xi=S(T_{\mathbf{R}}M)\otimes(\lambda\otimes\xi), \tag{4.7}$$

where $S(T_{\mathbf{R}}M)$ denotes spinors bundle and λ denotes the square root of $\det(TM)$. Then the Levi-Civita connection ∇^L associated to $g^{T_{\mathbf{R}}M}$ has a lift on $S(T_{\mathbf{R}}M)$, still denoted by ∇^L. Then by Theorem 2.3 in [Bi], we have:

$$2(\overline{\partial}+\overline{\partial}^\star)^2=-\sum_{i=1}^{2n}((\nabla^E_{e_i})^2-\nabla^E_{\nabla^L_{e_i}e_i})^2+\frac{k}{4}$$

$$+^c((\nabla^\xi)^2+\frac{1}{2}\mathrm{Tr}[(\nabla^{TM})^2]\otimes I_\xi)-\frac{\sqrt{-1}}{2}{}^c(\overline{\partial}\partial\widehat{\omega})-\frac{1}{8}||(\partial-\overline{\partial})\widehat{\omega}||^2, \tag{4.8}$$

where $E=-\frac{\sqrt{-1}}{4}(\partial-\overline{\partial})\widehat{\omega}$ and $\nabla^E=\nabla^L+S^E$ and $\langle S^E(X)Y,Z\rangle=2E(X,Y,Z)$ for $X,Y,Z\in\Gamma(T_{\mathbf{R}}M)$ and $\{e_1,\cdots,e_{2n}\}$ is an orthonormal basis associated to $(T_{\mathbf{R}}M,g)$. If we replace g by g^ε, then by (4.8), we have

$$2(\overline{\partial}+\overline{\partial}^\star_\varepsilon)^2=-\Delta^E_\varepsilon+\frac{k^\varepsilon}{4}+^c((\nabla^\xi)^2+\frac{1}{2}\mathrm{Tr}[(\nabla^{TM,\varepsilon})^2]\otimes I_\xi)_\varepsilon$$

$$-\frac{\sqrt{-1}}{2}{}^c(\overline{\partial}\partial\widehat{\omega}^\varepsilon)-\frac{1}{8}||(\partial-\overline{\partial})\widehat{\omega}^\varepsilon||^2_{g^\varepsilon}. \tag{4.9}$$

Next we will compute the adiabatic limits of some terms in (4.9). We assume

$$\omega_2=\sum_{1\leq i<j\leq 2n}f_{i,j}\overline{dx_i}\wedge\overline{dx_j};\ f_{i,j}=\omega_2(\frac{\partial}{\partial\overline{x_i}},\frac{\partial}{\partial\overline{x_j}}), \tag{4.10}$$

where $\{d\overline{x_1},\cdots,d\overline{x_{2n}}\}=\{dx_1,dy_1,\cdots,dx_n,dy_n\}$. By (4.2) and $\omega_2(X,Y)=0$ if X or Y in F, we know

$$f_{i,j}=0,\ \text{when } i \text{ or } j\leq 2p. \tag{4.11}$$

By direct computations and (4.11), we get

$$(\partial-\overline{\partial})\omega_2(f_i,f_j,f_k)=(\partial-\overline{\partial})\omega_2(f_i,f_j,h_k)=0; \tag{4.12}$$

$$\overline{\partial}\partial\omega_2(f_i,f_j,f_k,f_l)=\overline{\partial}\partial\omega_2(f_i,f_j,f_k,h_l)=0. \tag{4.13}$$

By (4.6) and (4.13), we have

$$
{}^c(\overline{\partial}\partial\widehat{\omega}^\varepsilon) = \sum_{1\leq i<j<k<l\leq 2p} (\overline{\partial}\partial\omega_1)(f_i, f_j, f_k, f_l)c(f_i)c(f_j)c(f_k)c(f_l)
$$

$$
+ \sum_{1\leq i<j\leq 2p}\sum_{1\leq k<l\leq 2q} (\overline{\partial}\partial\omega_2)(f_i, f_j, h_k, h_l)c(f_i)c(f_j)c(\sqrt{\varepsilon}h_k)c(\sqrt{\varepsilon}h_l) + O(\sqrt{\varepsilon}). \quad (4.14)
$$

By (4.6) (4.12) and the definition of the norm, we get

$$
||(\partial-\overline{\partial})\widehat{\omega}^\varepsilon||^2_{g^\varepsilon} = \sum_{1\leq i<j<k\leq 2p} [\sqrt{-1}(\partial-\overline{\partial})\omega_1(f_i, f_j, f_k)]^2
$$

$$
+ \sum_{1\leq i\leq 2p}\sum_{1\leq j<k\leq 2q} [\sqrt{-1}(\partial-\overline{\partial})\omega_2(f_i, h_j, h_k)]^2 + O(\varepsilon). \quad (4.15)
$$

Since ${}^c(\nabla^\xi)^2$ is independent of g^ε, we have

$$
{}^c(\nabla^\xi)^2 = \sum_{1\leq i<j\leq 2p} (\nabla^\xi)^2(f_i, f_j)c(f_i)c(f_j) + O(\sqrt{\varepsilon}). \quad (4.16)
$$

Next we consider ${}^c(\mathrm{Tr}[(\nabla^{TM,\varepsilon})^2])$. We extend $g^{T_{\mathbf{R}}M}$ (resp. $g^{T_{\mathbf{R}}M,\varepsilon}$) to an Hermitian metric $g^{T_{\mathbf{R}}M\otimes\mathbf{c}}$ (resp. $g^{T_{\mathbf{R}}M\otimes\mathbf{C},\varepsilon}$), then we get an Hermitian metric g^{TM} (resp. $g^{TM,\varepsilon}$) by restricting the $g^{T_{\mathbf{R}}M\otimes\mathbf{C}}$ (resp. $g^{T_{\mathbf{R}}M\otimes\mathbf{C},\varepsilon}$) to TM. Let $\{\frac{\partial}{\partial z_1}, \cdots, \frac{\partial}{\partial z_n}\}$ be a holomorphic local basis of TM and

$$
\widehat{H} = (H_{\alpha\beta})_{n\times n};\ \widehat{H}_{\alpha\beta} = g^{TM}(\frac{\partial}{\partial z_\alpha}, \frac{\partial}{\partial z_\beta}). \quad (4.17)
$$

Let $\widetilde{\omega}$ (resp. Ω) be the connection (resp. curvature) matrix associated to the Hermitian holomorphic connection $(TM, \nabla^{TM}, g^{TM})$ under the basis $\{\frac{\partial}{\partial z_1}, \cdots, \frac{\partial}{\partial z_n}\}$. Then

$$
\widetilde{\omega} = \partial\widehat{H}.\widehat{H}^{-1};\ \Omega = d\widetilde{\omega} - \widetilde{\omega}\wedge\widetilde{\omega}. \quad (4.18)
$$

By $T_{\mathbf{R}}M\otimes\mathbf{C} = F\otimes\mathbf{C}\oplus F^\perp\otimes\mathbf{C}$ and (M, F) is a complex foliation, we know

$$
TM = T^{1,0}F \oplus T^{1,0}F^\perp, \quad (4.19)
$$

where locally

$$
T^{1,0}F = \mathrm{span}_{\mathbf{C}}\{\frac{\partial}{\partial z_1}, \cdots, \frac{\partial}{\partial z_p}\} \quad (4.20)
$$

and $T^{1,0}F^\perp$ is the orthogonal complementary bundle of $T^{1,0}F$ in TM. We let

$$
\frac{\partial}{\partial z_r} = \sum_{j=1}^p a_{rj}\frac{\partial}{\partial z_j} + \sum_{s=1}^q b_{rs}\overline{e_s},\ p+1\leq r\leq n, \quad (4.21)
$$

where $\{\overline{e_1},\cdots,\overline{e_q}\}$ is a complex orthonormal basis of $T^{1,0}F^\perp$. Let $B=(b_{rs})_{q\times q}$, then

$$\begin{bmatrix} \frac{\partial}{\partial z_1} \\ \vdots \\ \frac{\partial}{\partial z_n} \end{bmatrix} = \begin{bmatrix} I_{p\times p} & 0 \\ \star & B \end{bmatrix} \begin{bmatrix} \frac{\partial}{\partial z_1} \\ \vdots \\ \frac{\partial}{\partial z_p} \\ \overline{e_1} \\ \vdots \\ \overline{e_s} \end{bmatrix}, \tag{4.22}$$

where B is invertible. Let

$$\widehat{H} = \begin{bmatrix} H_{pp} & H_{pq} \\ H_{qp} & H_{qq} \end{bmatrix}, \tag{4.23}$$

where $H_{pp}=[\widehat{H}_{\alpha\beta}]_{1\leq\alpha,\beta\leq p}$ is invertible. By (4.20),(4.21) and

$$g^{T_{\mathbf{R}}M\otimes\mathbf{c},\varepsilon} = g^{F\otimes\mathbf{C}} \oplus \frac{1}{\varepsilon} g^{F^\perp\otimes\mathbf{C}}, \tag{4.24}$$

then under the metric $g^{TM,\varepsilon}$, we have

$$\widehat{H}^\varepsilon = \begin{bmatrix} H_{pp} & H_{pq} \\ H_{qp} & \overline{H_{qq}} + \frac{1}{\varepsilon}\widehat{H_{qq}} \end{bmatrix}, \tag{4.25}$$

where $\widehat{H_{qq}} = B\overline{B}^t$ which is invertible. By the definition of the inverse of $\widehat{H}^\varepsilon$, we get

$$(\widehat{H}^\varepsilon)^{-1} = \frac{1}{\det H_{pp}\det\widehat{H}_{qq}+O(\varepsilon)} \begin{bmatrix} A_{11}+O(\varepsilon) & O(\varepsilon) \\ A_{21}\varepsilon+O(\varepsilon^2) & A_{22}\varepsilon+O(\varepsilon^2) \end{bmatrix}, \tag{4.26}$$

where

$$A_{11} = \det H_{pp}\det\widehat{H}_{qq}H_{pp}^{-1};\ A_{22} = \det H_{pp}\det\widehat{H}_{qq}\widehat{H_{qq}}^{-1}. \tag{4.27}$$

By (4.18),(4.25) and (4.26),

$$\widetilde{\omega}^\varepsilon = \frac{1}{\det H_{pp}\det\widehat{H}_{qq}} \begin{bmatrix} \partial H_{pp}.A_{11} & 0 \\ \partial H_{qp}.A_{11} + \partial\widehat{H_{qq}}.A_{21} & \partial\widehat{H_{qq}}.A_{22} \end{bmatrix} + O(\varepsilon). \tag{4.28}$$

Then by (4.18),(4.27),(4.28), we get

$$\mathrm{Tr}[(\nabla^{TM,\varepsilon})^2] = \mathrm{Tr}[\Omega^\varepsilon] = \mathrm{Tr}[\Omega^{T^{1,0}F}] + \mathrm{Tr}[\Omega^{T^{1,0}F^\perp}] + O(\varepsilon), \tag{4.29}$$

where

$$\begin{aligned} \Omega^{T^{1,0}F} &= d(\partial H_{pp}.H_{pp}^{-1}) - (\partial H_{pp}.H_{pp}^{-1})\wedge(\partial H_{pp}.H_{pp}^{-1}); \\ \Omega^{T^{1,0}F^\perp} &= d(\partial\widehat{H}_{qq}.\widehat{H}_{qq}^{-1}) - (\partial\widehat{H}_{qq}.\widehat{H}_{qq}^{-1})\wedge(\partial\widehat{H}_{qq}.\widehat{H}_{qq}^{-1}). \end{aligned} \tag{4.30}$$

By (4.29), we obtain

$$c\left(\mathrm{Tr}[(\nabla^{TM,\varepsilon})^2]\right) = \sum_{1\leq i<j\leq 2p}\left\{\mathrm{Tr}[\Omega^{T^{1,0}F}(f_i,f_j)] + \mathrm{Tr}[\Omega^{T^{1,0}F^\perp}(f_i,f_j)]\right\}c(f_i)c(f_j) + O(\sqrt{\varepsilon}). \tag{4.31}$$

Remark. In fact the term $O(\sqrt{\varepsilon})$ in (4.29) and (4.31) vanishes. We let

$$B_0 = \left[\begin{array}{cc} I_{p\times p} & 0 \\ \star & B \end{array} \right], \tag{4.32}$$

then

$$\widehat{H}^\varepsilon = B_0 \left[\begin{array}{cc} H_{pp} & 0 \\ 0 & \frac{1}{\varepsilon} \end{array} \right] \overline{B_0}^t;\ (\widehat{H}^\varepsilon)^{-1} = \overline{B_0^{-1}}^t \left[\begin{array}{cc} H_{pp}^{-1} & 0 \\ 0 & \varepsilon \end{array} \right] {B_0}^{-1}. \tag{4.33}$$

By (4.18), (4.33) and the trace property, we have

$$\mathrm{Tr}(\widetilde{\omega}^\varepsilon) = \mathrm{Tr}[\partial H_{pp}.H_{pp}^{-1} + \partial B_0 {B_0}^{-1} + \partial \overline{B_0} \overline{B_0}^{-1}]. \tag{4.34}$$

So the vanishing of the $O(\sqrt{\varepsilon})$ in (4.29) and (4.31) comes from (4.34) and

$$\mathrm{Tr}[\Omega^\varepsilon] = \overline{\partial}\mathrm{Tr}(\widetilde{\omega}^\varepsilon). \tag{4.35}$$

Let ϕ_{ij} be the (i,j)-component of $\phi \in \wedge^p T^{\mathbf{R},\star} M \otimes \mathbf{C}$ associated to the decomposition

$$\wedge^p T^{\mathbf{R},\star} M \otimes \mathbf{C} = \oplus_{i+j=p} \wedge^i F^\star \otimes \wedge^j F^{\perp,\star} \otimes \mathbf{C}. \tag{4.36}$$

We set

$$\begin{aligned} \Psi(x) = {} & \frac{(k^F + \Phi(\omega))(x)}{4} - || \frac{\sqrt{-1}}{2} {}^c[(\overline{\partial}\partial\omega_2)_{2,2}]||_x \\ & - ||\frac{\sqrt{-1}}{2} {}^c[(\overline{\partial}\partial\omega_1)_{4,0}]||_x - \frac{1}{8}||\sqrt{-1}[(\partial - \overline{\partial})\omega_2]_{1,2}||_x^2 \\ & -\frac{1}{8}||\sqrt{-1}[(\partial-\overline{\partial})\omega_1]_{3,0}||_x^2 - ||{}^c[(\nabla^\xi)^2]_{2,0}||_x - \frac{1}{2}||{}^c[\mathrm{Tr}(\Omega^{T^{1,0}F} + \Omega^{T^{1,0}F^\perp})]_{2,0}||_x, \end{aligned} \tag{4.37}$$

where $||.||_x$ denotes the norm of a linear operator acting on $\Big((\wedge T^{\star(0,1)} M \otimes \xi)_x, g_x\Big)$. As in the discussions in Section 2, we have

Theorem 4.1 *Let (M,F) be a compact complex foliation. If $\Psi > 0$, then the Euler number* $\mathrm{Eul}(\xi)$ *of the holomorphic bundle ξ vanishes.*

By Theorem 2.11 in [Bi], we have:

Corollary 4.2 *Let (M,F) be a compact complex foliation. If $\Psi > 0$ and $\overline{\partial}\partial(\omega_1 + \omega_2) = 0$, then there exists a $(2n-1)$-form τ such that*

$$\left\{ \widehat{A}\left(\frac{R^{-E}}{2\pi}\right) \exp\left(-\frac{1}{2\sqrt{-1}\pi}\mathrm{Tr}\left[\frac{(\nabla^{TM})^2}{2}\right]\right) \mathrm{Tr}\left[\exp\left(-\frac{(\nabla^\xi)^2}{2\sqrt{-1}\pi}\right)\right]\right\}^{\max} = d\tau. \tag{4.38}$$

5 A Kastler-Kalau-Walze type theorem for foliations

Several years ago, Connes made a challanging observation that the noncommutative residue (see [FGV] or [Wo]) of the $-2l+2$ power of the Dirac operator on $2l$ $(l \geq 2)$-dimensional spin manifolds was proportional to the Einstein-Hilbert action, which was called Kastler-Kalau-Walze Theorem now. In [Ka], Kastler gave a brute-force proof of this theorem. In [KW], Kalau and Walze proved this theorem by the normal coordinates way simultaneously. In [Ac], Ackermann gave a note on a new proof of this theorem by using heat kernel expansion.

Let (M^n, F) be a compact even dimensional foliation. In [Ko1], the author defined the tangential pseudodifferential operator algebra for foliations and a noncommutative residue on it. We assume M has spin leave. In order to give a Kastler-Kalau-Walze type theorem for foliations, it is natural to consider the noncommutative residue in [Ko1] of the $-n+2$ power of the Dirac operator along leave which is in the tangential pseudodifferential operator algebra. But the computations seems to be a little complicated , which comes from the tangential pseudodifferential calculas. Borrowing the idea in [LZ], we consider the sub-Dirac operator and the noncommutative residue on the classical pseudodifferential operator algebra on M. That is, we will compute

$$\lim_{\varepsilon\rightarrow 0}\mathrm{Res}(D_{F,\varepsilon}^{-n+2}),$$

where $D_{F,\varepsilon} = D_{F,\phi(F^\perp)\varepsilon}|_{\phi=1}$.

By (2.20) in [LZ], then

$$D_F^2 = -\triangle^F + \frac{k^{TM}}{4} + Q \tag{5.1}$$

where

$$\begin{aligned} Q = &\frac{1}{4}\sum_{i=1}^{p}\sum_{r,s,t=1}^{q}\langle R^{F^\perp}(f_i,h_r)h_t,h_s\rangle c(f_i)c(h_r)\widehat{c}(h_s)\widehat{c}(h_t) \\ &+\frac{1}{8}\sum_{i,j=1}^{p}\sum_{s,t=1}^{q}\langle R^{F^\perp}(f_i,f_j)h_t,h_s\rangle c(f_i)c(f_j)\widehat{c}(h_s)\widehat{c}(h_t) \\ &+\frac{1}{8}\sum_{s,t,r,l=1}^{q}\langle R^{F^\perp}(h_r,h_l)h_t,h_s\rangle c(h_r)c(h_l)\widehat{c}(h_s)\widehat{c}(h_t). \end{aligned} \tag{5.2}$$

By [Ac], we know that:

$$\mathrm{Res}(D_F^{-n+2}) = c_0\int_M \mathrm{Tr}_{S(F)\widehat{\otimes}\wedge(F^{\perp,\star})}(-\frac{k^{TM}}{12} - Q)d\mathrm{vol}_g, \tag{5.3}$$

where $c_0 = \frac{2}{(\frac{n}{2}-2)!\times(4\pi)^{\frac{n}{2}}}$. By the identity

$$d\mathrm{vol}_{g^\varepsilon} = \frac{1}{\varepsilon^{\frac{q}{2}}}d\mathrm{vol}_g, \tag{5.4}$$

we get

$$\lim_{\varepsilon\to 0}\varepsilon^{\frac{q}{2}}\mathrm{Res}(D_{F,\varepsilon}^{-n+2}) = c_0\int_M \lim_{\varepsilon\to 0}\mathrm{Tr}_{(S(F)\widehat{\otimes}\wedge(F^{\perp,\star}),g^\varepsilon)}\left(-\frac{k^{TM,\varepsilon}}{12}-Q^\varepsilon\right)\varepsilon^{\frac{q}{2}}d\mathrm{vol}_{g^\varepsilon}$$

$$= c_0\int_M \lim_{\varepsilon\to 0}\mathrm{Tr}_{g^\varepsilon}\left(-\frac{k^{TM,\varepsilon}}{12}-Q^\varepsilon\right)d\mathrm{vol}_g. \tag{5.5}$$

By (2.14) and $\mathrm{Tr}_{g^\varepsilon}(\widehat{c}(\sqrt{\varepsilon}h_s)) = \mathrm{Tr}_g(\widehat{c}(h_s))$, we have

$$\lim_{\varepsilon\to 0}\mathrm{Tr}_{g^\varepsilon}Q^\varepsilon = \mathrm{Tr}[\frac{1}{8}\sum_{i,j=1}^{p}\sum_{s,t=1}^{q}\langle \widehat{R}^{F^\perp}(f_i,f_j)h_t,h_s\rangle c(f_i)c(f_j)\widehat{c}(h_s)\widehat{c}(h_t)] = 0. \tag{5.6}$$

In the last equality of (5.6), we have used the identity

$$\mathrm{Tr}[c(f_i)c(f_j)] = \mathrm{Tr}[\widehat{c}(h_s)\widehat{c}(h_t)] = 0, \text{ for } i\neq j,\ s\neq t. \tag{5.7}$$

By (2.11) (5.5) (5.6), we have

$$\lim_{\varepsilon\to 0}\varepsilon^{\frac{q}{2}}\mathrm{Res}(D_{F,\varepsilon}^{-n+2}) = \widehat{c_0}\int_M [k^F+\Phi(\omega)]d\mathrm{vol}_g, \tag{5.8}$$

where $\widehat{c_0} = -\frac{c_0}{12}\mathrm{Rk}[S(F)\widehat{\otimes}\wedge(F^{\perp,\star})]$ and $\mathrm{Rk}[S(F)\widehat{\otimes}\wedge(F^{\perp,\star})]$ equals $2^{\frac{p}{2}+q}$ (resp. $2^{\frac{p-1}{2}+q}$) when p is even (resp. odd). So we get

Theorem 5.1 *Let (M^n,F) be a compact even-dimensional oriented foliation with spin leave and codimension q, and D_F be the sub-Dirac operator, then* $\lim_{\varepsilon\to 0}\varepsilon^{\frac{q}{2}}\mathrm{Res}(D_{F,\varepsilon}^{-n+2})$ *is proportional to* $\int_M [k^F+\Phi(\omega)]d\mathrm{vol}_g$.

Remark 1. When (M,F) is a fibre bundle with compact fibres, then $\Phi(\omega)=0$, so the $O(\frac{1}{\varepsilon^{\frac{q}{2}}})$ coefficient of $\mathrm{Res}(D_{F,\varepsilon}^{-n+2})$ is proportional to the sum of the gravity along fibre. Especially, when $p=0$, Theorem 5.1 is the classical Kastler-Kalau-Walze theorem.

Remark 2 By (2.32-2.35) in [LZ], we know that

$$k^{TM,\varepsilon} = k^F + \Phi(\omega) + a_1\varepsilon + a_2\varepsilon^2, \tag{5.9}$$

where a_1, a_2 are functions which are independent of ε. So, if $q=2$ (resp. 4), we will have that $\lim_{\varepsilon\to 0}\mathrm{Res}(D_{F,\varepsilon}^{-n+2})$ is proportional to $\int_M a_1 d\mathrm{vol}_g$ (resp. $\int_M a_2 d\mathrm{vol}_g$).

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